

ANALYTIC HYPOELLIPTICITY AT NON-SYMPLECTIC POISSON-TREVES STRATA FOR SUMS OF SQUARES OF VECTOR FIELDS

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ABSTRACT. We consider an operator P which is a sum of squares of vector fields with analytic coefficients. The operator has a non-symplectic characteristic manifold, but the rank of the symplectic form σ is not constant on $\text{Char } P$. Moreover the Hamilton foliation of the non symplectic stratum of the Poisson-Treves stratification for P consists of closed curves in a ring-shaped open set around the origin. We prove that then P is analytic hypoelliptic on that open set. And we note explicitly that the local Gevrey hypoellipticity for P is G^{k+1} and that this is sharp.

1. INTRODUCTION

The purpose of this paper is to study analytic hypoellipticity for some sums of squares of vector fields having a non trivial Poisson-Treves stratification. By this we mean that the stratification has non trivial deep strata.

After Treves introduced the Poisson stratification associated to a sum of squares of real analytic vector fields ([15], [4]), Hanges, [7], observed that if we look at analytic hypoellipticity in the sense of germs, there are operators with a non symplectic characteristic set which are analytic hypoelliptic in the sense of germs. Subsequently in [3] a class has been defined basically having the same properties of the Hanges operator and being analytic hypoelliptic in the sense of germs. It was also remarked that this fact is in no contradiction with Treves' conjecture roughly stating that a sum of squares is analytic hypoelliptic if and only if every stratum in its stratification is symplectic.

In all known examples we have analytic hypoellipticity in the sense of germs when the characteristic manifold is non symplectic and is actually a stratum of the stratification, which implies that the symplectic form

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$\sigma = d\xi \wedge dx$ has constant rank. Moreover the canonical 1-form $\omega = \xi dx$ does not vanish. This implies that the characteristic manifold has particularly simple microlocal models.

Moreover the bicharacteristic curves, i.e. the Hamilton leaves of the foliation, are closed curves foliating a given neighborhood of a characteristic point on which hypoellipticity in the sense of germs is obtained.

In this paper we study two different cases. The first is that of a sum of two squares for which the Poisson-Treves stratification has a symplectic “surface” stratum and a deeper non-symplectic stratum. The leaves of the Hamilton foliation are closed and foliate a certain open subset of the stratum:

$$P(t, x, D_t, D_x) = D_t^2 + [x_1 D_2 - x_2 D_1 + t^k (x_1 D_1 + x_2 D_2)]^2,$$

for $k \geq 2$ (see Section 2 for more details on its stratification).

It is well known that the above operator is G^{k+1} hypoelliptic and not better when $k \geq 2$, and is analytic hypoelliptic if $k = 1$.

We prove the following

Theorem 1.1. *Let $k \geq 2$ and P be as above. Let U be an open subset of \mathbb{R}^3 in the variables (t, x_1, x_2) projecting on an annulus of the form $r_1 < |x| < r_2$ and containing points where $t = 0$. Then P is analytic hypoelliptic (in the sense of germs) at points in $U \cap \Sigma_2$.*

For the proof we need to use ideas, specifically localizations of high order derivatives adapted to the problem at hand and less straightforward than those of [12], [13], introduced by Derridj and Tartakoff in [5].

Finally we discuss also another model operator which does not have closed orbits; in this case the orbits foliate an annulus in the x -variables, but have ω - and α -limit sets that are closed stationary orbits (see e.g. [8]).

In this case we have analytic regularity of the solution if the closed limit sets do not intersect the analytic wave front set of the solution. We do not attempt to prove any sort of analytic hypoellipticity in the sense of germs in this case, since the fact that the orbits are not closed does not seem to allow this.

2. PROOF OF THEOREM 1.1: SOME PREPARATIONS

For $k \in \mathbb{N}$, $k \geq 2$, let us consider the operator

$$(2.1) \quad P(t, x, D_t, D_x) = D_t^2 + [x_1 D_2 - x_2 D_1 + t^k(x_1 D_1 + x_2 D_2)]^2,$$

in the region $0 < r_1 \leq r \leq r_2$, $r = |x|$, $x = (x_1, x_2) \in \mathbb{R}^2$.

The Poisson-Treves stratification for P above is given by

$$(2.2) \quad \begin{aligned} \Sigma_1 &= \{\tau = 0, x_1 \xi_2 - x_2 \xi_1 + t^k(x_1 \xi_1 + x_2 \xi_2) = 0, t \neq 0\}; \\ \Sigma_2 &= \{\tau = 0, t = 0, x_1 \xi_2 - x_2 \xi_1 = 0, \xi = (\xi_1, \xi_2) \neq 0\} \\ \Sigma_j &= \Sigma_2, \quad \text{for } j \leq k, \\ \Sigma_{k+1} &= \{0\}, \end{aligned}$$

where the last equation means that Σ_{k+1} is just the zero section of $R^*\mathbb{R}^3$.

We explicitly remark that Σ_1 is a symplectic submanifold of codimension 2, while Σ_2 is not symplectic. Moreover $\Sigma_1 \cup \Sigma_2 = \text{Char } P$.

Let us denote by

$$(2.3) \quad X_1 = D_t$$

$$(2.4) \quad \begin{aligned} X_2 &= x_1 D_2 - x_2 D_1 + t^k(x_1 D_1 + x_2 D_2) \\ &= D_\theta + t^k D_r = D_\theta + R \end{aligned}$$

so that

$$(2.5) \quad P = X_1^2 + X_2^2.$$

with

$$(2.6) \quad [X_1, X_2] = kt^{k-1}R, \text{ and } [R, X_j] = 0, j = 1, 2.$$

We have the a priori estimate

$$(2.7) \quad \|v\|_{1/(k+1)}^2 + \|X_1 v\|^2 + \|X_2 v\|^2 \leq C \{ |\langle P v, v \rangle| + \|v\|^2 \}, \quad v \in C_0^\infty.$$

where $\|\cdot\|$ denotes the L^2 -norm in $\mathbb{R}_t \times \mathbb{R}_x^2$.

2.1. The general scheme. Since the operator is subelliptic, the solution will be in C^∞ . Additionally, since for $t \neq 0$, the characteristic manifold of P is symplectic, we know the solution is analytic for $t \neq 0$. With a localizing function $\varphi(r)$ to be made precise below (but of Ehrenpreis type), and exploiting the maximality of the *a priori* estimate satisfied by P , we will study $\|\varphi X^{p+1} u\|^2$, each occurrence of X being X_1 or X_2 .

Using the *a priori* estimate effectively will require moving one X to the left of φ but this will not present a problem in the ensuing recursion.

We will immediately be led to estimate the bracket $|\langle [P, \varphi X^p]u, \varphi X^p u \rangle|$. Upon iteration, using 2.6, we arrive, after at most p iterations of the *a priori* estimate, to terms of the form

$$(2.8) \quad \underline{C}^p \|(X)\varphi^{(p+1)}u\|^2 \text{ or } \underline{C}^p C p!! \|(X)\varphi R^{p/2}u\|^2$$

and of course all the intermediate terms with some derivatives on φ , some powers of p , and some powers of R , all with the generic bounds

$$C C^p p^a \|(X)\varphi^{(b)} R^a u\| \text{ with } p \sim b + 2a.$$

Here $p!! = p(p-2)(p-4)\dots$, the value of C may change from line to line but always independently of u and the order of differentiation, and underlining a coefficient indicates the number of terms of the form which follows that occur. Finally, writing (X) means that an $X = X_1$ or X_2 may or may not be present.

When all X 's have been consumed in this way, we may no longer iterate effectively, and we must turn our attention to pure powers of R , suitably localized. This will require a new localizing function and a construction we denote $(R^q)_\psi$ reminiscent of [12] and [13], or more precisely [?], which requires a special vector field M which commutes especially well with both X_1 and X_2 , namely reproducing X_1 or generating R .

2.2. The vector field M and the localization. We are fortunate to have a ‘good’ vector field M at our disposal which reproduces R by bracketing with X_2 : with

$$(2.9) \quad M = \frac{t}{k} D_t,$$

we have

$$(2.10) \quad [M, X_2] = R$$

As localizing functions we shall use a nested family of Ehrenpreis-type functions as used by the second author in [12], [13]. Given $N \in \mathbb{N}$, the band $r \in (r_1, r_2)$ will contain $\log_2 N$ nested subbands, $\Omega_k = \{r : r_{1_k} \leq r \leq r_{2_k}\}$, $r_{1_0} = r_1, r_{2_0} = r_2, k \leq \log_2 N$, with

$$(2.11) \quad d_k = r_{1_k} - r_{1_{k-1}} = r_{2_k} - r_{2_{k-1}} = (r_2 - r_1) \frac{1}{4k^2}$$

(so that $\sum d_k \leq r_2 - r_1$) and functions $\varphi_k \equiv 1$ on Ω_k and supported in Ω_{k+1} , such that with a constant $C = C_{r_2-r_1}$,

$$(2.12) \quad |\varphi_k^{(\ell)}(r)| \leq (C/d_k)^{\ell+1} N_k^\ell \text{ for } \ell \leq N_k = N/2^{k-1}.$$

The functions φ_k , but not the constant C , depend on the choice of N . In fact, we shall double the number of these functions, for technical reasons, $\varphi_1, \tilde{\varphi}_1, \varphi_2, \tilde{\varphi}_2, \dots$ with φ_j and $\tilde{\varphi}_j$ satisfying the same growth estimates.

We note in passing, and will use later, that the growth estimates (2.12) imply the (weaker) growth estimates

$$(2.13) \quad |\varphi_k^{(\ell)}(r)| \leq (C'/d_k)^{N_k^\ell+1} \ell! \text{ for } \ell \leq N_k = N/2^{k-1}.$$

(using the fact that $y^{1/y} \leq e$ for $y \geq 1$.)

Given the definition of M above, and for $p \in \mathbb{R}$ large and $j \in \{0, 1, \dots, p\}$ we define the expressions

$$(2.14) \quad N_j = \sum_{j'=0}^j a_{j'}^j \frac{M^{j'}}{j'!},$$

where the $a_{j'}^j$ denote rational numbers satisfying properties that shall be made precise below which optimize commutation relations.

Finally we define our localizing operator, which is equal to R^p where $\varphi \equiv 1$. We let

$$(2.15) \quad R_\varphi^p = \sum_{j=0}^p \varphi^{(j)} N_j R^{p-j} = \sum_{j=0}^p (R^j \varphi) N_j R^{p-j}.$$

2.3. The Commutation Relations for R_φ^p . For two vector fields Z and \tilde{Z} we shall frequently use the formula

$$(2.16) \quad [Z^j, \tilde{Z}] = \sum_{k=1}^j \binom{j}{k} \text{ad}_Z^k(\tilde{Z}) Z^{j-k},$$

where

$$\text{ad}_Z(\tilde{Z}) = [Z, \tilde{Z}], \quad \text{ad}_Z^2(\tilde{Z}) = [Z, [Z, \tilde{Z}]]$$

et cetera.

2.4. The bracket $[X_2, R_\varphi^p]$. We first compute the commutator of X_2 with N_j . We have

$$[X_2, N_j] = \sum_{j'=1}^j a_{j'}^j [X_2, \frac{M^{j'}}{j'!}] = -R^k \sum_{j'=1}^j a_{j'}^j \sum_{\ell=1}^{j'} \frac{1}{\ell!} \frac{M^{j'-\ell}}{(j'-\ell)!} R,$$

since

$$[X_2, M] = -t^k R.$$

We seek to find coefficients $a_{j'}^j$, so that

$$[X_2, N_j] = -t^k N_{j-1} R,$$

which will ensure that the bracket $[X_2, R_\varphi^p]$ is free of the (poorly controlled) vector field R (see below). Using (??), the necessary condition is that the $a_{j'}^j$ must satisfy

$$(2.17) \quad \sum_{j'=1}^j \sum_{\ell=1}^{j'} a_{j'}^j \frac{1}{\ell!} \frac{M^{j'-\ell}}{(j'-\ell)!} = \sum_{j_1=1}^{j-1} a_{j_1}^{j-1} \frac{M^{j_1}}{j_1!},$$

or

$$(2.18) \quad \sum_{s=1}^{j-\ell} a_{\ell+s}^j \frac{1}{s!} = a_\ell^{j-1},$$

for $\ell = 0, 1, \dots, j-1$.

We shall come back to condition (2.18) later; for the time being we may conclude the following

Lemma 2.1. *With the coefficients $a_{j'}^j$ chosen as above,*

$$[X_2, N_j] = -t^k N_{j-1} R,$$

for every $j \in \mathbb{N}$.

Proposition 2.1.

$$[X_2, R_\varphi^p] = t^k \varphi^{(p+1)} N_p.$$

Proof. Using the above Lemma we have, for these $a_{j'}^j$,

$$\begin{aligned} [X_2, R_\varphi^p] &= t^k \sum_{j=0}^p \varphi^{(j+1)} N_j R^{p-j} - t^k \sum_{j=1}^p \varphi^{(j)} N_{j-1} R^{p+1-j} \\ &= t^k \varphi^{(p+1)} N_p + t^k \sum_{j=1}^p \varphi^{(j)} N_{j-1} R^{p-(j+1)} - t^k \sum_{j=1}^p \varphi^{(j)} N_{j-1} R^{p+1-j} \end{aligned}$$

$$= t^k \varphi^{(p+1)} N_p.$$

□

2.5. The bracket $[X_1, R_\varphi^p]$. First remark that

$$(2.19) \quad \text{ad}_M^\ell(X_1) = \left(-\frac{1}{k}\right)^\ell X_1,$$

for every $\ell \in \mathbb{N}$. Therefore,

$$[X_1, N_j] = \sum_{j'=1}^j a_{j'}^j [X_1, \frac{M^{j'}}{j'!}] = -X_1 \sum_{j'=1}^j \sum_{\ell=1}^{j'} a_{j'}^j \left(-\frac{1}{k}\right)^\ell \frac{1}{\ell!} \frac{M^{j'-\ell}}{(j'-\ell)!},$$

so that we have

$$(2.20) \quad [X_1, R_\varphi^p] = \sum_{j=1}^p \varphi^{(j)}(-X_1) \sum_{j'=1}^j \sum_{\ell=1}^{j'} a_{j'}^j \left(-\frac{1}{k}\right)^\ell \frac{1}{\ell!} \frac{M^{j'-\ell}}{(j'-\ell)!} R^{p-j}.$$

Our next goal is to prove the following lemma:

Lemma 2.2. *For every $j \in \mathbb{N}$ and $\ell \in \{1, \dots, j\}$ there exist real constants δ_s , $s = 0, \dots, j-2$, such that*

$$(2.21) \quad \sum_{h=1}^{j-\ell} a_{\ell+h}^j \left(-\frac{1}{k}\right)^h \frac{1}{h!} = \sum_{h=1}^{j-\ell} \delta_{j-\ell-h} a_\ell^{\ell+h-1}.$$

The above Lemma has an easy consequence:

Lemma 2.3. *For every $j \in \mathbb{N}$ there exist real constants γ_s , $s = 0, \dots, j-1$, such that*

$$(2.22) \quad \sum_{j'=1}^j \sum_{\ell=1}^{j'} a_{j'}^j \left(-\frac{1}{k}\right)^\ell \frac{1}{\ell!} \frac{M^{j'-\ell}}{(j'-\ell)!} = \sum_{s=0}^{j-1} \gamma_{j-s} N_s.$$

Proof of Lemma 2.3. The identity (2.22) can be restated as

$$\sum_{s=0}^{j-1} \sum_{j'=s+1}^j a_{j'}^j \left(-\frac{1}{k}\right)^{j'-s} \frac{1}{(j'-s)!} \frac{M^s}{s!} = \sum_{s=0}^{j-1} \sum_{h=0}^s \gamma_{j-s} a_h^s \frac{M^h}{h!}$$

or

$$\sum_{s=0}^{j-1} \sum_{j'=s+1}^j a_{j'}^j \left(-\frac{1}{k}\right)^{j'-s} \frac{1}{(j'-s)!} \frac{M^s}{s!} = \sum_{\ell=0}^{j-1} \sum_{s=\ell}^{j-1} \gamma_{j-s} a_\ell^s \frac{M^\ell}{\ell!}.$$

From which we get

$$\sum_{j'=\ell+1}^j a_{j'}^j \left(-\frac{1}{k}\right)^{j'-\ell} \frac{1}{(j'-\ell)!} = \sum_{s=\ell}^{j-1} \gamma_{j-s} a_\ell^s.$$

Now the latter identity can be rewritten as

$$\sum_{h=1}^{j-\ell} a_{\ell+h}^j \left(-\frac{1}{k}\right)^h \frac{1}{h!} = \sum_{h=1}^{j-\ell} \delta_{j-\ell-h} a_\ell^{\ell+h-1},$$

for any $\ell = 1, \dots, j-1$, and this is in the statement of Lemma 2.2. \square

Proof of Lemma 2.2. In order to prove Lemma 2.2 we must analyse the recurrence relation (2.18):

$$\sum_{h=1}^{j-\ell} a_{\ell+h}^j \frac{1}{h!} = a_\ell^{j-1},$$

for $\ell = 0, 1, \dots, j-1$.

Another way of rewriting the above relation is the following:

$$(2.23) \quad \begin{bmatrix} 1 & \frac{1}{2!} & \cdots & \frac{1}{(j-1)!} & \frac{1}{j!} \\ 0 & 1 & \cdots & \frac{1}{(j-2)!} & \frac{1}{(j-1)!} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \frac{1}{2!} \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1^j \\ \vdots \\ a_j^j \end{bmatrix} = \begin{bmatrix} a_0^{j-1} \\ \vdots \\ a_{j-1}^{j-1} \end{bmatrix}.$$

Note that on the left hand side there are no terms of the form a_0^j , which means that we are free to choose those coefficients. We shall choose $a_0^0 = 1$ for the sake of simplicity, leaving the others undetermined.

We point out that the matrix in the above formula is clearly invertible and that it can be written as

$$I_j + \frac{1}{2!} J_j + \frac{1}{3!} J_j^2 + \cdots + \frac{1}{j!} J_j^{j-1} = \int_0^1 e^{tJ_j} dt,$$

where J_j denotes the standard $j \times j$ Jordan matrix

$$J_j = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Using, for example, formula (2.23) we may easily see that, inverting the matrix, we obtain

$$(2.24) \quad \begin{bmatrix} a_1^j \\ \vdots \\ a_j^j \end{bmatrix} = \begin{bmatrix} c_0 & c_1 & \cdots & c_{j-2} & c_{j-1} \\ 0 & c_0 & \cdots & c_{j-3} & c_{j-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & c_0 & c_1 \\ 0 & 0 & \cdots & 0 & c_0 \end{bmatrix} \begin{bmatrix} a_0^{j-1} \\ \vdots \\ a_{j-1}^{j-1} \end{bmatrix},$$

where $c_0 = 1$, $c_1 = -\frac{1}{2!}$ and the other c_m can be computed by a triangular relation. In particular, using the structure of the matrix, we obtain that

$$(2.25) \quad \begin{bmatrix} a_{\ell+1}^j \\ \vdots \\ a_j^j \end{bmatrix} = \begin{bmatrix} c_0 & c_1 & \cdots & c_{j-\ell-2} & c_{j-\ell-1} \\ 0 & c_0 & \cdots & c_{j-\ell-3} & c_{j-\ell-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & c_0 & c_1 \\ 0 & 0 & \cdots & 0 & c_0 \end{bmatrix} \begin{bmatrix} a_\ell^{j-1} \\ \vdots \\ a_{j-1}^{j-1} \end{bmatrix},$$

for $\ell = 0, 1, \dots, j-1$.

Another way of writing the above identity is

$$(2.26) \quad a_{\ell+h}^j = \sum_{s=h}^{j-\ell} c_{s-h} a_{\ell-1+s}^{j-1},$$

for $h = 1, 2, \dots, j-\ell$.

Iterating, we get

$$a_{\ell-t+h}^{j-t} = \sum_{s=h}^{j-\ell} c_{s-h} a_{\ell-t-1+s}^{j-t-1},$$

for $t = 0, 1, \dots, j-\ell-1$.

Let us now fix an $h \in \{1, \dots, j - \ell\}$. Then we have

$$\begin{aligned}
a_{\ell+h}^j &= \sum_{s_1=h}^{j-\ell} c_{s_1-h} a_{\ell-1+s_1}^{j-1} \\
&= \sum_{s_1=h}^{j-\ell} \sum_{s_2=s_1}^{j-\ell} c_{s_1-h} c_{s_2-s_1} a_{\ell-2+s_2}^{j-2} \\
&= \sum_{s_1=h}^{j-\ell} \sum_{s_2=s_1}^{j-\ell} \cdots \sum_{s_h=s_{h-1}}^{j-\ell} c_{s_1-h} c_{s_2-s_1} \cdots c_{s_h-s_{h-1}} a_{\ell-h+s_h}^{j-h}
\end{aligned}$$

The latter sum can be written as

$$\begin{aligned}
a_{\ell+h}^j &= c_0^h a_\ell^{j-h} + \sum_{s_1=h}^{j-\ell} \sum_{s_2=s_1}^{j-\ell} \cdots \sum_{\substack{s_h=s_{h-1} \\ s_h > h}}^{j-\ell} c_{s_1-h} c_{s_2-s_1} \cdots c_{s_h-s_{h-1}} a_{\ell-h+s_h}^{j-h} \\
&= c_0^h a_\ell^{j-h} + \sum_{s_1=h}^{j-\ell} \sum_{s_2=s_1}^{j-\ell} \cdots \sum_{\substack{s_h=s_{h-1} \\ s_h > h}}^{j-\ell} \sum_{s_{h+1}=s_h}^{j-\ell} c_{s_1-h} \cdots c_{s_{h+1}-s_h} a_{\ell-h-1+s_{h+1}}^{j-h-1}
\end{aligned}$$

The latter sum allows us to compute the coefficient of a_ℓ^{j-h-1} , by picking all terms for which one of the s_j is equal to $h+1$, for $j = 1, 2, \dots, h+1$.

Iterating this procedure, i.e. using the recursion relation until we obtain a coefficient a_*^* where the lower index is equal to ℓ , we may express the coefficient $a_{\ell+h}^j$ as a linear combination of a_ℓ^* ; the above formulas show that we may actually write

$$(2.27) \quad a_{\ell+h}^j = \sum_{\sigma=0}^{j-\ell-h} \alpha_{j-\ell-\sigma} a_\ell^{\ell+\sigma},$$

for $h = 1, 2, \dots, j - \ell$.

We point out explicitly that up to this point we have only used the recurrence relation (2.18). Let us now denote by A_h the collection of real numbers $A_h = (-k^{-1})^h h!^{-1}$. Then it is evident that

$$\sum_{h=1}^{j-\ell} a_{\ell+h}^j A_h = \sum_{\sigma=0}^{j-\ell-1} \delta_{j-\ell-\sigma} a_\ell^{\ell+\sigma},$$

where $\delta_{j-\ell-\sigma} = \sum_{h=1}^{j-\ell-\sigma} A_h$, and this is the statement of the Lemma. \square

Lemma 2.4 (See [5] and [10]). *Let us consider the recurrence relation (2.18):*

$$\sum_{s=1}^{j-\ell} a_{\ell+s}^j \frac{1}{s!} = a_{\ell}^{j-1}.$$

Setting

$$(2.28) \quad a_{\ell}^j = \frac{1}{(j-\ell)!} \left(\left[\frac{t}{e^t - 1} \right]^{j+1} \right)^{(j-\ell)} (0),$$

we obtain a solution of the above recurrence satisfying the boundary conditions $a_j^j = 1$ and $a_0^j = (-1)^j$, $j \geq 0$. Moreover this is the only power series with rational coefficients satisfying (2.18) and the above boundary conditions.

Proof. By a simple computation we have

$$\begin{aligned} a_{\ell}^{j-1} &= \frac{1}{(j-1-\ell)!} \left(\left(\frac{t}{e^t - 1} \right)^j \right)^{(j-1-\ell)} (0) \\ &= \frac{1}{(j-1-\ell)!} \left(\left(\frac{t}{e^t - 1} \right)^{j+1} \frac{e^t - 1}{t} \right)^{(j-1-\ell)} (0) \\ &= \sum_{h=0}^{j-1-\ell} \frac{1}{(j-1-\ell-h)!} \left(\left(\frac{t}{e^t - 1} \right)^{j+1} \right)^{(j-1-\ell-h)} (0) \frac{1}{(h+1)!} \\ &= \sum_{p=1}^{j-\ell} \frac{1}{(j-1-p)!} \left(\left(\frac{t}{e^t - 1} \right)^{j+1} \right)^{(j-\ell-p)} (0) \frac{1}{p!} \\ &= \sum_{p=1}^{j-\ell} a_{\ell+p}^j \frac{1}{p!}. \end{aligned}$$

Where we used the fact that

$$\frac{1}{h!} \left(\frac{e^t - 1}{t} \right)^{(h)} (0) = \frac{1}{(h+1)!}.$$

Moreover we have $a_j^j = 1$ for every $j \geq 0$. As for the other boundary condition, first we remark that

$$a_0^j = \frac{1}{j!} \left(\left[\frac{t}{e^t - 1} \right]^{j+1} \right)^{(j)} (0),$$

i.e. a_0^j is the coefficient of t^j in the power series of $Q(t)$,

$$Q(t) = \left(\frac{t}{e^t - 1} \right)^{j+1}.$$

Thus

$$a_0^j = \frac{1}{2i\pi} \int_{\gamma} \left(\frac{1}{e^z - 1} \right)^{j+1} dz,$$

where γ is a smooth curve encircling the origin in \mathbb{C} .

Changing variables $w = e^z - 1$, so that the origin is mapped to the origin and γ is mapped to another smooth curve encircling the origin that we still denote by γ , we have

$$a_0^j = \frac{1}{2i\pi} \int_{\gamma} w^{-(j+1)} (w+1)^{-1} dz = (-1)^j.$$

The uniqueness is proved in [10]. This ends the proof of the lemma. \square

As a consequence of the preceding Lemmas we may now state the

Proposition 2.2. *The commutator of X_1 with the localizing operator R_{φ}^p has the form*

$$(2.29) \quad [X_1, R_{\varphi}^p] = -X_1 \sum_{\ell=0}^{p-1} \delta_{\ell} R_{\varphi^{(\ell+1)}}^{p-\ell-1},$$

where $\varphi^{(j)}$ denotes the j -th derivative $(r\partial_r)^j \varphi$ and $\delta_{\ell} = \sum_1^{\ell} \frac{1}{k^h h!} \leq 1$.

From Lemma 2.4 we have the

Corollary 2.1. *For every $j \geq 0$ and $\ell \in \{0, \dots, j\}$ we have*

$$|a_{\ell}^j| \leq c^j,$$

for a suitable universal positive constant c .

Proof. From (2.28) we have that

$$a_{\ell}^j = \frac{1}{2i\pi} \int_{\gamma} \left(\frac{t}{e^t - 1} \right)^{j+1} t^{-(j-\ell+1)} dt,$$

where γ is a circle of fixed radius around the origin. Since the function under the integral sign may be estimated by a positive constant (depending on the radius of γ) raised to the power j , the corollary follows. \square

3. PROOF OF THEOREM 1.1

In this section we prove that P is analytic hypoelliptic in any open set of the form $\Omega = \{(t, x) \in \mathbb{R}^3 \mid r_1 < |x| < r_2, t \in (-\delta, \delta)\}, \delta > 0$.

The maximal estimate may be restated to allow X to appear to the right or left of the localizing function (where $\psi' = X\psi$):

$$\begin{aligned}
 (3.1) \quad & \|\psi X^p u\|_{\frac{1}{k}}^2 + \|X\psi X^p u\|^2 + \|\psi X^{p+1} u\|^2 \lesssim \\
 & \lesssim |\langle P\psi X^p u, \psi X^p u \rangle| + \|\psi' X^p u\|^2 \lesssim \\
 & \lesssim |\langle \psi X^p P u, \psi X^p u \rangle| + |\langle [X^2, \psi X^p] u, \psi X^p u \rangle| + \|\psi' X^p u\|^2
 \end{aligned}$$

Now

$$\begin{aligned}
 |\langle [X^2, \psi X^p] u, \psi X^p u \rangle| & \leq |\langle [X, \psi X^p] u, X\psi X^p u \rangle| + |\langle [X, \psi X^p] X u, \psi X^p u \rangle| \\
 & \lesssim |\langle \psi' X^p u, X\psi X^p u \rangle| + \underline{p} |\langle t^{k-1} \psi R X^{p-1} u, X\psi X^p u \rangle| \\
 & \quad + |\langle \psi X^p X u, \psi' X^p u \rangle| + \underline{p} |\langle t^{k-1} \psi X^p R u, \psi X^p u \rangle| \\
 & \leq \varepsilon \{ \|X\psi X^p u\|^2 + \|\psi X^{p+1} u\|^2 \} + C_\varepsilon \{ \|\psi' X^p u\|^2 + (\underline{p} \|\psi X^{p-1} R u\|)^2 \}
 \end{aligned}$$

where we have freely exchanged ψ and ψ' on the two sides of the inner product when no derivatives intervened. Note that $[X, R] = 0$.

In all,

$$\begin{aligned}
 & \|\psi X^p u\|_{\frac{1}{k}}^2 + \|X\psi X^p u\|^2 + \|\psi X^{p+1} u\|^2 \\
 & \lesssim \|\psi X^p P u\|^2 + \|\psi' X^p u\|^2 + (\underline{p} \|\psi R X^{p-1} u\|)^2
 \end{aligned}$$

Iterating this inequality until there remain no X 's on the right,

$$\begin{aligned}
 (3.2) \quad & \|\psi X^p u\|_{\frac{1}{k}}^2 + \|X\psi X^p u\|_{L^2}^2 + \|\psi X^{p+1} u\|_{L^2}^2 \\
 & \leq C^p \left\{ \sup_{j+2d \leq p} \{ p^d \|\psi^{(j)} R^d X^{p-j-2d} P u\|_{L^2}^2 \} + \sup_{j+2d=p} (p^d \|\psi^{(j)} R^d u\|_{L^2})^2 \right\}.
 \end{aligned}$$

The first term on the right can be estimated directly (even taken to be zero, using the Cauchy-Kowalevski theorem). For the second, we will take the localizing function out of the norm and introduce one of the $(R^d)_{\tilde{\psi}} \equiv R^d$ on the support of ψ . Thus for such $\tilde{\psi}$, and taking $Pu = 0$ for simplicity,

$$\begin{aligned}
 (3.3) \quad & \|\psi X^p u\|_{\frac{1}{k}} + \|X\psi X^p u\|_{L^2} + \|\psi X^{p+1} u\|_{L^2} \leq \sup_{j+2d=p} p^d \sup |\psi^{(j)}| \| (R^d)_{\tilde{\psi}} u \|_{L^2}
 \end{aligned}$$

For convenience we recall the bracket relations and the few important definitions (for generic φ):

$$\begin{aligned} [X_1, R_\varphi^b] &= -X_1 \sum_{\ell=0}^{b-1} \delta_\ell R_{\varphi(\ell+1)}^{b-\ell-1}, \quad |\delta_\ell| \leq 1 \\ [X_2, R_\varphi^b] &= t^k \varphi^{(b+1)} N_b. \\ N_b &= \sum_{b'=0}^b a_{b'}^b \frac{M^{b'}}{b'!}, \quad M = \frac{t}{k} D_t, \quad |a_{b'}^b| \leq c^b. \end{aligned}$$

As above, we use the *a priori* estimate, but now on $v = (R^d)_\varphi u$:

$$\begin{aligned} (3.4) \quad & \| (R^d)_\varphi u \|_{\frac{1}{k}}^2 + \| X(R^d)_\varphi u \|^2 + \| (R^d)_\varphi Xu \|^2 \lesssim \\ & \lesssim |\langle P(R^d)_\varphi u, \psi(R^d)_\varphi u \rangle| + \| [X, (R^d)_\varphi] u \|^2 \lesssim \\ & \lesssim |\langle (R^d)_\varphi Pu, (R^d)_\varphi u \rangle| + |\langle [X^2, (R^d)_\varphi] u, (R^d)_\varphi u \rangle| + \| [X, (R^d)_\varphi] u \|^2. \end{aligned}$$

Again, taking $Pu = 0$, and expanding $[X^2, (R^d)_\varphi] = X[X, (R^d)_\varphi] + [X, (R^d)_\varphi]X$, we find, as before, with a weighted Schwarz inequality and integrating by parts one $X = -X^*$,

$$\begin{aligned} (3.5) \quad & \| (R^d)_\varphi u \|_{\frac{1}{k}}^2 + \| X(R^d)_\varphi u \|^2 + \| (R^d)_\varphi Xu \|^2 \\ & \lesssim |\langle [X, (R^d)_\varphi] Xu, (R^d)_\varphi u \rangle| + \| [X, (R^d)_\varphi] u \|^2 \end{aligned}$$

Now on the right, when $X = X_1$, the result, as we saw above, still has an X_1 , which we integrate by parts in the case of the inner product:

$$\begin{aligned} (3.6) \quad & |\langle [X_1, (R^d)_\varphi] X_1 u, (R^d)_\varphi u \rangle| + \| [X_1, (R^d)_\varphi] u \|^2 \\ & \leq \varepsilon \| X_1 (R^d)_\varphi u \|^2 + C_\varepsilon \sum_{d_1=1}^d \| (R^{d-d_1})_{\varphi(d_1)} X_1 u \|^2 \end{aligned}$$

On the other hand, when $X = X_2$, we have nearly pure powers of tD_t , which it will be necessary to convert into pure powers of $X_1 = D_t$ (from which we started, but, we note, of at most half the order).

Proposition 3.1.

$$(tD_t)^j = \sum_{\ell=1}^j B_\ell^j t^\ell D_t^\ell$$

where

$$B_\ell^j = \sum_{m=0}^{\ell-1} \frac{(-1)^m (\ell-m)^{j-1}}{m! (\ell-m-1)!} = \sum_{m=0}^{\ell-1} \frac{(-1)^m (\ell-m)^j}{m! (\ell-m)!}$$

so that for all v , pointwise,

$$\frac{|(tD_t)^j v|}{j!} \leq C^j \sum_{\ell=1}^j \frac{|t^\ell D_t^\ell v|}{\ell!}$$

and hence in $|t| < 1$,

$$|N_b v| = \left| \sum_{b'=0}^b a_{b'}^b \frac{M^{b'}}{b'!} v \right| = \left| \sum_{b'=0}^b \left(\frac{1}{k}\right)^{b'} a_{b'}^b \frac{(tD_t)^{b'}}{b'!} v \right| \leq C^b \sup_{b' \leq b} \left| \frac{X_1^{b'} v}{b'!} \right|$$

The particular expression for the coefficients B_ℓ^j is proved by induction and can be understood by a kind of over-counting/under-counting argument.

Thus for X_2 ,

$$\begin{aligned} (3.7) \quad & | \langle [X_2, (R^d)_\varphi] X_2 u, (R^d)_\varphi u \rangle | + \| [X_2, (R^d)_\varphi] u \|^2 \\ & \leq | \langle t^k \varphi^{(d+1)} N_d X_2 u, (R^d)_\varphi u \rangle | + \| t^k \varphi^{(d+1)} N_d u \|^2 \\ & \leq C^d \sup_{d' \leq d} \| \varphi^{(d+1)} \frac{X_1^{d'}(X_2)u}{d'!} \|^2 + \varepsilon \| (R^d)_\varphi u \|^2 \end{aligned}$$

or in all,

$$\begin{aligned} (3.8) \quad & \| (R^d)_\varphi u \|_{\frac{1}{k}}^2 + \| X(R^d)_\varphi u \|^2 + \| (R^d)_\varphi X u \|^2 \\ & \leq C \sum_{d_1=1}^d \| (R^{d-d_1})_{\varphi^{(d_1)}} X_1 u \|^2 + C^d \sup_{d' \leq d} \| \varphi^{(d+1)} \frac{X_1^{d'}(X_2)u}{d'!} \|^2 \end{aligned}$$

(i.e., with or without X_2 in the last term). Iterating on the first term on the right, eventually only the last term survives:

$$\begin{aligned} (3.9) \quad & \| (R^d)_\varphi u \|_{\frac{1}{k}}^2 + \| X(R^d)_\varphi u \|^2 + \| (R^d)_\varphi X u \|^2 \\ & \leq C^d \sup |\varphi^{(d+1)}| \sup_{d' \leq d} \| \tilde{\varphi} \frac{X^{d'+1} u}{(d'+1)!} \|^2 \end{aligned}$$

for any $\varphi, \tilde{\varphi}$ with $\tilde{\varphi} \equiv 1$ on the support of φ .

Recalling the previous bound

$$\begin{aligned} (3.10) \quad & \| \psi X^p u \|_{\frac{1}{k}}^2 + \| X \psi X^p u \|_{L^2}^2 + \| \psi X^{p+1} u \|_{L^2}^2 \\ & \leq \sup_{j+2d=p} p^d \sup |\psi^{(j)}| \| (R^d)_{\tilde{\varphi}} u \|_{L^2}^2, \end{aligned}$$

valid for any $\tilde{\psi} \equiv 1$ on the support of ψ , we have the choice of starting with X 's, reducing the order by half, introducing $(R^d)_\varphi$ and iterating that until we are back to X 's or start with $(R^d)_\varphi$, reduce to X 's until

they bracket to yield pure R 's at half the order. In either order, after one full cycle, we need a new localizing function each time $(R^d)_\varphi$ is put together. Thus in starting with N derivatives to estimate, after $\log_2 N$ full cycles, the number of free derivatives on u will be only a bounded number.

For definiteness, we follow the cycle starting with powers of X 's, and introduce for a moment the new norms

$$|||\psi, X^p, u||| = \|\psi X^p u\|_{\frac{1}{k}} + \|X\psi X^p u\|_{L^2} + \|\psi X^{p+1} u\|_{L^2}$$

and

$$|||(R^d)_\varphi, u||| = \|(R^d)_\varphi u\|_{\frac{1}{k}} + \|X(R^d)_\varphi u\|_{L^2} + \|(R^d)_\varphi X u\|_{L^2},$$

so that the above may be written

$$(3.11) \quad |||(R^d)_\varphi, u||| \leq C^d \sup_{d' \leq d} \frac{1}{(d' + 1)!} |||\varphi^{(d+1)}, X^{d'+1}, u|||$$

for any φ and

$$(3.12) \quad |||\psi, X^p, u||| \leq \sup_{j+2d=p} p^d \sup |\psi^{(j)}| |||(R^d)_{\tilde{\psi}}, u|||,$$

Thus we start with $\psi = \varphi_1$ (the first in the sequence of precisely nested localizing functions (cf. (2.12)) for a fixed $N = N_1 \in \mathbb{N}$:

$$\begin{aligned} \frac{\|X^{N_1} u\|_{L^2(\Omega_1)}}{N_1!} &\leq \frac{|||\varphi_1, X^{N_1}, u|||}{N_1!} \leq \sup_{N_2 \leq \tilde{N}_1 \leq N_1} \frac{|||\varphi_1, X^{\tilde{N}_1}, u|||}{\tilde{N}_1!} \\ &\leq \sup_{\substack{\ell_1 + 2\delta_1 = \tilde{N}_1 \\ N_2 \leq \tilde{N}_1 \leq N_1}} \frac{\tilde{N}_1^{\delta_1} \sup |\varphi_1^{(\ell_1)}| |||(R^{\delta_1})_{\tilde{\varphi}}, u|||}{\tilde{N}_1!} \\ &\leq \sup_{\substack{\ell_1 + 2\delta_1 = \tilde{N}_1 \\ N_2 \leq \tilde{N}_1 \leq N_1}} \frac{\left(\frac{C}{d_1}\right)^\ell \tilde{N}_1^{\ell_1 + \delta_1} |||(R^{\delta_1})_{\tilde{\varphi}}, u|||}{\tilde{N}_1!} \end{aligned}$$

with any $\tilde{\varphi} \equiv 1$ near the support of φ_1 .

Now there is some freedom in the choice of $\tilde{\varphi}$, since all that we have required is that it be one on the support of φ_1 , and we pick the largest index k consistent with δ_1 , i.e., $N_k \geq \delta_1 \geq N_{k+1}$ and $k \geq 2$ since

$\ell_1 + 2\delta_1 = N_1$.) Thus with $\tilde{\varphi} = \varphi_k$ and together with the other estimate:

$$(3.13) \quad |||(R^\delta)_{\varphi_k}, u||| \leq C^\delta \sup |\varphi_k^{(\delta+1)}| \sup_{\delta' \leq \delta} \frac{|||\tilde{\varphi}_k, X^{\delta'+1}, u|||}{(\delta' + 1)!},$$

we arrive at

$$\begin{aligned} & \frac{\|X^{N_1} u\|_{L^2(\Omega_1)}^2}{N_1^{N_1}} \leq \sup_{N_2 \leq \tilde{N}_1 \leq N_1} \frac{|||\tilde{\varphi}_1, X^{\tilde{N}_1}, u|||}{\tilde{N}_1^{\tilde{N}_1}} \\ & \leq \sup_{\substack{\ell+2\delta=\tilde{N}_1 \\ N_2 \leq \tilde{N}_1 \leq N_1+1}} \frac{\left(\frac{C}{d_1}\right)^\ell \tilde{N}_1^{\ell+\delta} C^\delta \sup |\varphi_k^{(\delta+1)}|}{\tilde{N}_1^{\tilde{N}_1}} \sup_{\substack{N_{k+1} \leq \tilde{N}_k \leq N_{k+1} \\ k \geq 2}} \frac{|||\tilde{\varphi}_k, X^{\tilde{N}_k}, u|||}{\tilde{N}_k^{\tilde{N}_k}} \\ & \leq \sup_{\substack{\ell+2\delta=\tilde{N}_1 \\ N_2 \leq \tilde{N}_1 \leq N_1}} \frac{\left(\frac{C}{d_1}\right)^\ell \tilde{N}_1^{\ell+\delta} C^\delta \left(\frac{C}{d_k}\right)^{\delta+1} N_k^{(\delta+1)}}{\tilde{N}_1^{\tilde{N}_1}} \sup_{\substack{N_{k+1} \leq \tilde{N}_k \leq N_{k+1} \\ k \geq 2}} \frac{|||\tilde{\varphi}_k, X^{\tilde{N}_k}, u|||}{\tilde{N}_k^{\tilde{N}_k}} \\ & \leq C^{\tilde{N}_1} \sup_{\ell+2\delta=\tilde{N}_1} \frac{d_1^{-\ell} d_k^{-(\delta+1)}}{2^{\ell+\delta} 2^{k(\delta+1)}} \sup_{\substack{N_{k+1} \leq \tilde{N}_k \leq N_{k+1} \\ k \geq 2}} \frac{|||\tilde{\varphi}_k, X^{\tilde{N}_k}, u|||}{\tilde{N}_k^{\tilde{N}_k}}. \end{aligned}$$

Now the expressions in the first supremum increase as k decreases, bounded by $d_1^{-(N_1+1)}/2^{N_1+1}$. Iteration will introduce another coefficient bounded by $C^{N_2} d_2^{-(N_2+1)}/2^{2(N_2+1)}$, then next by $C^{N_3} d_3^{-(N_3+1)}/2^{3(N_3+1)}$. Since

$$\frac{d_k^{-(N_k+1)}}{2^{k(N_k+1)}} = C^{N_k} \frac{(k^2)^{N_k+1}}{(2^k)^{N_k+1}},$$

iteration at most $\log_2 N$ times will lead to a product

$$\Pi_{k=1}^{\log_2 N} C^{N_k} \left(\frac{k^2}{2^k}\right)^{\frac{N}{2^k}+1} \leq (C')^N$$

times a constant depending only on the first few derivatives of u in the largest open set encountered.

This yields the analyticity of u in the smallest open set since all estimates are uniform in N .

4. THE CASE OF NON CLOSED BICHARACTERISTICS WITH NON TRIVIAL LIMIT SET

We want to study a model of the form

$$P(t, x, D_t, D_x) = D_t^2 + X_2^2$$

where

$$X_2 = g_1(x)g_2(x)[x_1D_2 - x_2D_1 + \mu(x_1D_1 + x_2D_2) + t^k(x_1D_1 + x_2D_2)],$$

$$g_1(x) = |x|^2 - a^2,$$

$$g_2(x) = b^2 - |x|^2,$$

with $0 < a < b$ in the open set $a < |x| < b$ and $\mu > 0$ is a given constant.

The characteristic set of P in the above mentioned region is $\text{Char}(P) = \{\tau = 0, x_1\xi_2 - x_2\xi_1 + \mu(x_1\xi_1 + x_2\xi_2) + t^k(x_1\xi_1 + x_2\xi_2) = 0\}$.

As for the Poisson stratification of P we have

$$\Sigma_1 = \{\tau = 0, x_1\xi_2 - x_2\xi_1 + \mu(x_1\xi_1 + x_2\xi_2) + t^k(x_1\xi_1 + x_2\xi_2) = 0, t \neq 0\};$$

$$\Sigma_2 = \{\tau = t = 0, x_1\xi_2 - x_2\xi_1 + \mu(x_1\xi_1 + x_2\xi_2) = 0, x_1\xi_1 + x_2\xi_2 \neq 0\};$$

$$\Sigma_{k+1} = \{0\},$$

i.e. the zero section of $R^*\mathbb{R}^3$ over the above specified region.

Evidently, since $\text{codim } \Sigma_2 = 3$, Σ_2 (or rather its connected components) is not a symplectic submanifold of $R^*\mathbb{R}^3$.

Let us take a look at the Hamilton foliation of Σ_2 . Define

$$(4.1) \quad A = \begin{bmatrix} \mu & 1 \\ -1 & \mu \end{bmatrix},$$

so that

$$(4.2) \quad \Sigma_2 = \{\tau = 0 = t, \langle x, A\xi \rangle = 0\},$$

where $x = (x_1, x_2)$ and $\xi = (\xi_1, \xi_2)$, $\xi \neq 0$ and $\langle x, \xi \rangle \neq 0$.

Then we know that $\langle x, A\xi \rangle \equiv 0$ on every leaf in Σ_2 , i.e. on every integral curve of the Hamilton field of $\langle x, A\xi \rangle$ issued from a point in Σ_2 .

The Hamilton system is

$$(4.3) \quad \begin{aligned} \dot{x} &= g_1(x)g_2(x)^t Ax \\ \dot{\xi} &= -g_1(x)g_2(x)A\xi. \end{aligned}$$

We easily see that, because of the structure of the matrix A , we have

$$\begin{aligned}\frac{1}{2}d_t|x|^2 &= g_1(x)g_2(x)\frac{\mu}{2}|x|^2 \\ \frac{1}{2}d_t|\xi|^2 &= -g_1(x)g_2(x)\frac{\mu}{2}|\xi|^2,\end{aligned}$$

so that both the spatial and the covariable projections of the bicharacteristics are logarithmic spirals. Moreover the spatial projection spirals between the two asymptotic circles $g_i(x) = 0$, $i = 1, 2$, which are stationary orbits of the first two equations in (4.3).

We point out that $d_t\langle x, \xi \rangle \equiv 0$, so that $\langle x, \xi \rangle$ is constant along the orbits and that once the first two equations in (4.3) are solved the second couple—i. e. the covariable projection—is easy:

$$\xi(t) = \exp \left[- \int_0^t g_1(x(s))g_2(x(s))ds \right] A \xi_0,$$

where ξ_0 is its initial data.

We may apply to the operator P Theorem 4.2 in [11] and conclude that, if γ_0 denotes a segment of a bicharacteristic curve in Σ_2 , then either $\gamma_0 \subset WF_a(u)$ or $\gamma_0 \cap WF_a(u) = \emptyset$, where u is a solution of $Pu \in C^\omega$ in some open set.

Let now U be an open set in \mathbb{R}^3 projecting onto an annulus of the form $a < |x| < b$ in the x -variables. By iteratively applying the above mentioned theorem one can prove the following

Theorem 4.1. *Let u be a distribution such that $Pu \in C^\omega(U)$, U being defined as above. Then if both circles $g_i(x) = 0$ do not intersect $WF_a(u)$ we have that $u \in C^\omega(U)$.*

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